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SIMULATION OF MULTI-VARIABLE CONVERTERS USING THE LINEAR INTERPOLATION METHOD

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Annotation. In this work, based on the theory of barycentric coordinates and simplexes, a linear interpolation method is proposed for modeling and controlling the operation of multiparameter converters. It has been determined that the linear interpolation method minimizes the structural diagram of a computing device, which makes it possible to more accurately determine the metrological characteristics of multiparameter measuring transducers and offer effective methods and means for processing primary measurement information. A theorem has been proven about a linear interpolating polynomial of a function of many variables, which will allow us to judge the property of linearization of multidimensional quantities from both qualitative and instrumental points of view, and a theorem that helps determine an accurate estimate of the interpolation error.

Keywords: multiparameter measuring transducers, approximation, linear interpolation, modeling, barycentric coordinates, error, multidimensional function.

Introduction

Multiparameter converters are designed to measure physical and chemical parameters and alarm systems in automatic monitoring and control systems of objects in the electric power industry, technological processes and production and to issue output signals in the form of DC power and voltage,
as well as non-electrical quantities converted into DC electrical signals or active resistance. The tasks of conversion and ensuring the accuracy of calculations are essential to ensure the required metrological characteristics of multiparameter measuring and information systems [1, 2, 3, 4].

When assessing the errors in determining vector quantities, it is necessary to proceed from a given criterion and the assumption that the solution to the system of equations describing the vector value converter is unique. In practice, the most significant contribution to the error in estimating vector quantities is made by the error $\delta_a$ in the approximation of the characteristics of the converter of a vector value into an information signal, as well as the error $\delta_p$ in solving the system of equations. [5, 6]

The approximation of the transformation function of a multiparameter converter should provide maximum approximation accuracy with a minimum number of nodes (experimental data); be simple enough so that the technical implementation of the computational processing algorithm is economical; provide the ability to quickly and sufficiently test multi-parameter converters. [7, 8, 9, 10-15]

**Research Methods and Received Results**

Based on the theory of barycentric coordinates and simplexes, we consider the proposed method of linear interpolation of a function of many variables.

Let us turn to a multiparameter converter or device, the input of which is supplied with an $n$-dimensional value $\vec{x} \in E^n$, and at the output there is a one-dimensional value $u$, which is a function of $n$ variable coordinates of the input, i.e.

$$u = u(\vec{x}) = u(x_1, x_2, \ldots, x_n)$$

(1)

Let’s consider the problem of linear interpolation of a function of many variables. Let us first present the functional dependence of the output one-dimensional value $u(1)$ on the vector value $\vec{x}$ in the form of a linear form, i.e. as a linear polynomial in a function of several variables

$$u(\vec{x}) = (\vec{p}, \vec{x}) + p_o = \sum_{i=0}^{n} p_i x_i + p_o$$

(2)

To determine the coefficients $p_i, 0 \leq i \leq n$ of this linear form, it is necessary to apply not one signal to the input of a multiparameter system (or device), as in the one-dimensional case, but several multidimensional signals, i.e. $(n+1)$ vectors $\vec{x}^s \in E^n, 0 \leq s \leq n$, forming a simplex.

Note that the set of input vectors $\{\vec{x}^s \in E^n, 0 \leq s \leq n\}$ forms a simplex if the system of vectors $\{\vec{x}^s - \vec{x}^0, 1 \leq s \leq n\}$ is linearly independent.

As an example of such simplicial input vectors, we can take the zero vector $\vec{x}^0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ with the corresponding one-dimensional output value $u_0 = u(\vec{x}^0) = u(\vec{0}) = 0$, as well as the following multidimensional input vectors (test signals):

$$\vec{x}^1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{x}^2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \vec{x}^n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

with corresponding one-dimensional outputs $u_1, u_2, \ldots, u_n, \ldots, u_n$.

Coefficients $p_i$ of linear form (2) are determined from solving a system of linear equations

$$L(\vec{x}^s) = p_0 + \sum_{i=0}^{n} p_i x_i^s = u, 0 \leq s \leq n$$

(3)

which can be represented in vector-matrix notation as follows:

$$\vec{A} \vec{p} = \vec{u}$$

(4)

where $A$ is a matrix of dimension $(n+1) \times (n+1)$.
CHEMICAL TECHNOLOGY. CONTROL AND MANAGEMENT.

\[
A = \begin{bmatrix}
1 & x_1^0 & ... & x_i^0 & ... & x_n^0 \\
1 & x_1^1 & ... & x_i^1 & ... & x_n^1 \\
... & ... & ... & ... & ... & ... \\
1 & x_1^s & ... & x_i^s & ... & x_n^s \\
... & ... & ... & ... & ... & ... \\
1 & x_1^n & ... & x_i^n & ... & x_n^n
\end{bmatrix}
= \begin{bmatrix}
1 & \bar{x}_0 \\
1 & \bar{x}_1 \\
... & ... \\
1 & \bar{x}_s \\
... & ... \\
1 & \bar{x}_n
\end{bmatrix}
\]

(5)

\[
\vec{p}, \vec{u} \text{ — column vectors of dimension (n+1):}
\]

\[
\vec{p} = \begin{bmatrix}
p_0 \\
p_1 \\
... \\
p_s
\end{bmatrix}, \quad \vec{u} = \begin{bmatrix}
u_0 \\
u_1 \\
... \\
u_n
\end{bmatrix}
\]

(6)

It should be noted that, based on the property of simplicity of input quantities, matrix A is non-degenerate [11, 12, 13].

To assess the error of the linear interpolation method, we consider its main qualitative properties, which also characterize the capabilities of the hardware implementation of this algorithm.

We will assume that all multidimensional inputs (input quantities) will be inside the simplex, \( \bar{x} \in \text{simplex} \{ \bar{x}^s; 0 \leq s \leq n \}. \) In other words, for an arbitrary multidimensional input quantity the following relation holds true:

\[
\bar{x} = \sum_{s=0}^{n} \lambda_s \bar{x}^s
\]

(7)

In this case, the coefficients of the linear combination (6) \( \lambda_s(\bar{x}), 0 \leq s \leq n \) will satisfy the relation

\[
\sum_{s=0}^{n} \lambda_s(\bar{x}) = 1
\]

(8)

If this multidimensional quantity belongs to a simplex, then in addition to (7) it is true that all \( \lambda_s(\bar{x}), 0 \leq s \leq n \) are not negative \( \lambda_s(\bar{x}) \leq 0 \).

In mathematics and some other areas of physics, the coefficients \( \lambda_s(\bar{x}), 0 \leq s \leq n \) are called barycentric coordinates relative to the vertices of the simplex. [14, 15]

Taking into account relations (7) and (6), with a priori given multidimensional value \( \bar{x} \) and fixed vertices of the simplex \( \{ \bar{x}^s; 0 \leq s \leq n \} \), the barycentric coordinates \( \lambda_s = \lambda_s(\bar{x}), 0 \leq s \leq n \) can be determined from solving a system of linear algebraic equations of (n+1) order, which is written in vector-matrix form as follows:

\[
y' \vec{\lambda} = \vec{\bar{x}}
\]

(9)

where \( y \) is a matrix of dimension (n+1)×(n+1):

\[
y = \begin{bmatrix}
1 & 1 & ... & 1 \\
\bar{x}_0^0 & \bar{x}_1^1 & ... & \bar{x}_s^s \\
... & ... & ... & ... \\
\bar{x}_0^n & \bar{x}_1^n & ... & \bar{x}_n^n
\end{bmatrix} = A'
\]

\( \vec{\lambda}, \vec{\bar{x}} \) — column vectors of dimension (n+1):
Having solved system (8), we obtain
\[ \tilde{\lambda} = y^{-1} \tilde{x} = (A^T)^{-1} \tilde{x} = Z \tilde{x} \]  
(11)
where the Z-matrix is the inverse of the y-matrix.
In coordinate form, relation (11) can be written as follows:
\[ \lambda_s = \sum_{i=1}^{n} z_{is} x_i + z_{0s}, 0 \leq s \leq n \]  
(12)
From relation (7) it is easy to study changes in barycentric coordinates depending on the input multidimensional value \( \tilde{x} \), i.e.
\[ \sum_{s=0}^{n} \nabla \lambda_s (\tilde{x}) = 0 \]  
(13)
where \( \nabla \lambda_s (\tilde{x}) \) - gradient of the s-th barycentric coordinate.
To characterize the change in the barycentric coordinate in the direction from the multidimensional value \( \tilde{x} \) to one of the fixed vertices of the simplex (Fig. 1 illustrates such a change with \( n=2 \) for the direction to one of the test signals \( \tilde{x}^s \)), the following lemma is valid.

**Figure 1.**

**Lemma.** The rate of change of an arbitrary \( v \)-th barycentric coordinate \( \lambda_v(\tilde{x}); 0 \leq v \leq n \) in the direction \( \tilde{x}^s - \tilde{x} \); \( 0 \leq s \leq n \) is characterized by the relation
\[ (\nabla \lambda_v(\tilde{x}), \tilde{x}^s - \tilde{x}) = \delta_{vs} - \lambda_v(\tilde{x}) \]  
(14)
where \( \delta_{vs} \) - Kronecker symbol.

**Proof:** From formula (12) it is clear that \( \frac{\partial \lambda_v}{\partial x_m} = z_{mv} \). Describing scalar product (14) in more detail, we have
\[
\left[ \nabla \lambda_v(\tilde{x}), \tilde{x}^s - \tilde{x} \right] = \sum_{m=1}^{n} \frac{\partial \lambda_v}{\partial x_m} (x_m^s - x_m) = \\
= \sum_{m=1}^{n} z_{mv} x_v^s + z_{ov} - \sum_{m=1}^{n} z_{mv} x_m - z_{ov} 
\]  
(15)
Due to the fact that the minuend \( \sum_{m=1}^{n} z_{mv} x_v^s + z_{ov} \) is the product of the \( v \)-th row of the matrix \( Z \) on the \( S \)-th column of the matrix \( y \) and \( Zy = E \), then
\[
\sum_{m=1}^{n} z_{mv}x_m^S + z_{ov} = \delta_{vs}.
\]  
(16)

According to relation (12) the subtrahend

\[- \left( \sum_{m=1}^{n} z_{mv}x_m + z_{ov} \right) = -\hat{\lambda}_v(\tilde{x})
\]  
(17)

Thus, relation (14) is obtained, which indicates the dependence of the rate of change of the barycentric coordinate \(\lambda_v(\tilde{x})\) (Fig. 1) towards the \(s\)-th vertex of the base multidimensional value \(\tilde{x}^S\).

Using the results obtained, we will consider one of the approaches to determining the linearized multidimensional transformation function. We can point to the following theorem about the linear interpolating polynomial of a function of many variables, which will allow us to judge the property of linearization of multidimensional quantities from the output readings corresponding to the input multidimensional base vectors (test signals) from both qualitative and hardware points of view.

**Theorem 1.** A linear interpolating polynomial of a function of several variables of the form

\[L(\tilde{x}) = (\tilde{p}, \tilde{x}) + p_0\]

can be based on the relation

\[L(\tilde{x}) = \sum_{v=0}^{n} \hat{\lambda}_v(\tilde{x})u_v
\]  
(18)

where \(u_v = u(\tilde{x}^v)\) is the output reading of the device which corresponds to the \(v\)-th base input vector (test signal) \(\tilde{x}^v\).

**Proof:** Note that \(\lambda_v(\tilde{x}^v) = \delta_{vs}\). Due to the fact that \(L(\tilde{x})\) is a linear form, \(\lambda_v(\tilde{x})\) for any \(0 \leq v \leq n\) are linear forms, and any linear form in \(n\)-dimensional space is determined by its \(N=(n+1)\) values, namely

\[L(\tilde{x}^v) = u_v; 0 \leq v \leq n \hat{\lambda}_v(\tilde{x}^0) = 0; \hat{\lambda}_v(\tilde{x}^1) = 0; \ldots; \hat{\lambda}_v(\tilde{x}^v) = 0; \ldots; \hat{\lambda}_v(\tilde{x}^n) = 0,
\]

which completes the proof of the theorem.

An important task in studying the computational process of linear interpolation of multidimensional functions is the problem of accurately estimating the interpolation error. The following theorem allows us to solve this problem.

**Theorem 2.** The following exact estimates of the error of the linear interpolation method are valid:

\[L(\tilde{x}) - u(\tilde{x}) = (\tilde{p}, \tilde{x}) + p_0 - u(\tilde{x}) = \frac{1}{2} \sum_{v=0}^{n} \left[ H(\tilde{y}^v)(\tilde{x}^v - \tilde{x}), \tilde{x}^v - \tilde{x} \right] \hat{\lambda}_v(\tilde{x})
\]  
(19)

\[\nabla L(\tilde{x}) - \nabla u(\tilde{x}) = p - \nabla u(\tilde{x}) = \frac{1}{2} \sum_{v=0}^{n} \left[ H(\tilde{y}^v)(\tilde{x}^v - \tilde{x}), \tilde{x}^v - \tilde{x} \right] \nabla \hat{\lambda}_v(\tilde{x}),
\]  
(20)

where \(\tilde{y}^v = \tilde{x}^v + \theta_v(\tilde{x} - \tilde{x}^v); 0 < \theta_v < 1; 0 \leq v \leq n, H(\tilde{y}^v)\) is Hessian matrix at the intermediate point \(\tilde{y}^v\), i.e.

\[H(\tilde{y}^v) = \partial^2 u(\tilde{y}^v) / \partial x_i \partial x_j; 0 \leq i, j \leq n
\]  
(21)

**Proof:** To prove the theorem, we expand \(u_v\) into a Taylor series. As a result we get

\[u_v = u(\tilde{x}^v) \approx u(\tilde{x}) + (\nabla u(\tilde{x}), \tilde{x}^v - \tilde{x}) + \frac{1}{2} \left[ H(\tilde{y}^v)(\tilde{x}^v - \tilde{x}), \tilde{x}^v - \tilde{x} \right]
\]  
(22)

According to Theorem 1

\[L(\tilde{x}) = \sum_{v=0}^{n} \hat{\lambda}_v(\tilde{x})u(\tilde{x}^v) = \sum_{v=0}^{n} \hat{\lambda}_v(\tilde{x}) \left[ u(\tilde{x}) + (\nabla u(\tilde{x}), \tilde{x}^v - \tilde{x}) + \frac{1}{2} \left[ H(\tilde{y}^v)(\tilde{x}^v - \tilde{x}), \tilde{x}^v - \tilde{x} \right] \right]
\]  
(23)
Taking into account that \( \sum_{\nu=0}^{n} \lambda_{\nu}(\tilde{x}) = 1 \), and taking into account that \( \sum_{\nu=0}^{n} \lambda_{\nu}(\tilde{x}) \tilde{x}^{\nu} = \tilde{x} \), we finally get

\[
L(\tilde{x}) = u(\tilde{x}) + \frac{1}{2} \sum_{\nu=0}^{n} \left[ H(\tilde{y}^{\nu})(\tilde{x}^{\nu} - \tilde{x}), \tilde{x}^{\nu} - \tilde{x} \right] \lambda_{\nu}(\tilde{x})
\]

(24)

which proves the first part of the theorem.

To prove the second part of the theorem, we also use **Theorem 1**.

Differentiating relation (19), we obtain

\[
\nabla L(\tilde{x}) = p = \sum_{\nu=0}^{n} \nabla \lambda_{\nu}(\tilde{x}) u(\tilde{x}^{\nu})
\]

(25)

Using the Taylor expansion again, we obtain

\[
\nabla L(\tilde{x}) = \sum_{\nu=0}^{n} \nabla \lambda_{\nu}(\tilde{x}) \left[ u(\tilde{x}) + \left( \nabla u(\tilde{x}), \tilde{x}^{\nu} - \tilde{x} \right) \right] +
\]

\[
+ \frac{1}{2} \left[ H(\tilde{y}^{\nu})(\tilde{x}^{\nu} - \tilde{x}), \tilde{x}^{\nu} - \tilde{x} \right] \lambda_{\nu}(\tilde{x}) +
\]

\[
+ \sum_{\nu=0}^{n} \left( \nabla u(\tilde{x}), \tilde{x}^{\nu} - \tilde{x} \right) \nabla \lambda_{\nu}(\tilde{x}) + \frac{1}{2} \left[ H(\tilde{y}^{\nu})(\tilde{x}^{\nu} - \tilde{x}), \tilde{x}^{\nu} - \tilde{x} \right] \nabla \lambda_{\nu}(\tilde{x})
\]

Taking into account (13)

\[
\sum_{\nu=0}^{n} \nabla \lambda_{\nu}(\tilde{x}) u(\tilde{x}^{\nu}) = 0
\]

Expression (26) will be somewhat simplified and will have the form

\[
\nabla L(\tilde{x}) = \sum_{\nu=0}^{n} \left( \nabla u(\tilde{x}), \tilde{x}^{\nu} - \tilde{x} \right) \nabla \lambda_{\nu}(\tilde{x}) + \frac{1}{2} \left[ H(\tilde{y}^{\nu})(\tilde{x}^{\nu} - \tilde{x}), \tilde{x}^{\nu} - \tilde{x} \right] \nabla \lambda_{\nu}(\tilde{x})
\]

(27)

Next, we will show that

\[
\sum_{\nu=0}^{n} \left( \nabla u(\tilde{x}), \tilde{x}^{\nu} - \tilde{x} \right) \nabla \lambda_{\nu}(\tilde{x}) \equiv \nabla u(\tilde{x}).
\]

Let us introduce the notation

\[
\vec{r} = \sum_{\nu=0}^{n} \left( \nabla u(\tilde{x}), \tilde{x}^{\nu} - \tilde{x} \right) \nabla \lambda_{\nu}(\tilde{x})
\]

(28)

and consider the scalar products of a vector \( \vec{r} \) with \( n \)-linearly independent vectors \( \tilde{x}^{s} - \tilde{x} ; 1 \leq s \leq n \).

Taking into account the lemma, we get

\[
\langle \vec{r}, \tilde{x}^{s} - \tilde{x} \rangle = \left[ \sum_{\nu=0}^{n} \left( \nabla u(\tilde{x}), \tilde{x}^{\nu} - \tilde{x} \right) \nabla \lambda_{\nu}(\tilde{x}) \right] \tilde{x}^{s} - \tilde{x} \rangle = \sum_{\nu=0}^{n} \left( \nabla u(\tilde{x}), \tilde{x}^{\nu} - \tilde{x} \right) \delta_{s\nu} - \lambda_{s}(\tilde{x}) =
\]

\[
= \left( \nabla u(\tilde{x}), \tilde{x}^{s} - \tilde{x} \right) - \sum_{\nu=0}^{n} \lambda_{s}(\tilde{x}) \left( \nabla u(\tilde{x}), \tilde{x}^{\nu} - \tilde{x} \right) =
\]

\[
= \left( \nabla u(\tilde{x}), \tilde{x}^{s} - \tilde{x} \right) - \left( \nabla u(\tilde{x}), \sum_{\nu=0}^{n} \lambda_{\nu}(\tilde{x}) \tilde{x}^{\nu} - \sum_{\nu=0}^{n} \lambda_{\nu}(\tilde{x}) \tilde{x} \right) =
\]

\[
= \left( \nabla u(\tilde{x}), \tilde{x}^{s} - \tilde{x} \right) - (\nabla u(\tilde{x}), \tilde{x} - \tilde{x}) = (\nabla u(\tilde{x}), \tilde{x}^{s} - \tilde{x})
\]

Thus,

\[
\nabla L(\tilde{x}) = p = \nabla u(\tilde{x}) + \frac{1}{2} \sum_{\nu=0}^{n} \left[ H(\tilde{y}^{\nu})(\tilde{x}^{\nu} - \tilde{x}), \tilde{x}^{\nu} - \tilde{x} \right] \nabla \lambda_{\nu}(\tilde{x}).
\]

This completes the proof of the second part of the main theorem, characterizing the quality of the process of linearization of a multidimensional function.

The resulting error estimate for the linear interpolation method is accurate.

In practice, when considering an \( n \)-dimensional parallelepiped, at least \( N_{1}^{*} = 2^{n} \) preliminary measurements of a multidimensional value (basic vectors) will be required.

In our case (the case of the simplicial approach), a minimum of points is required – only \( N_{1} = (n + 1) \) preliminary measurements of a multidimensional quantity.
Conclusion

Thus, as a result of using the simplicial approach, a significant gain is obtained by reducing the required number of measurements (reference or test signals) of a multidimensional quantity. The gain is especially noticeable when the dimension of a multidimensional quantity increases. We emphasize that in practice this gain is realized not only in the form of a reduction in measurement time, but sometimes (for example, for gas analytical systems) and in the form of savings in materials (reference gas mixtures).

References